

38

AD 665029

**A DECISION THEORY APPROACH  
TO ACCEPTANCE TESTING**

By

E. D. SIMMONS and C. E. WISLER  
Operations Research Division

31 January 1968

FEB 16 1968

THIS DOCUMENT HAS BEEN APPROVED FOR PUBLIC RELEASE  
AND SALE; ITS DISTRIBUTION IS UNLIMITED.



**NAVAL MISSILE CENTER**

**Point Mugu, California**

Reproduced by the  
CLEARINGHOUSE  
for Federal Scientific & Technical  
Information Springfield Va 22151

36

# NAVAL MISSILE CENTER

AN ACTIVITY OF THE NAVAL AIR SYSTEMS COMMAND

C. J. RICKETTS, CAPT USN

*Commander*

D. F. SULLIVAN

*Technical Director*

This report describes work accomplished under Local Project L-2492, Operations Research.

Dr. W. M. Simpson, Head, Operations Research Division; W. L. MacDonald, Chief Engineer, Weapons, Weapon Program Management Department; and CAPT R. C. Thatcher, Jr., Head Weapons Program Management Department, have reviewed this report for publication.

Stamp: 10/10/77, with a signature and checkboxes.

THIS REPORT HAS BEEN PREPARED PRIMARILY FOR TIMELY PRESENTATION OF INFORMATION. ALTHOUGH CARE HAS BEEN TAKEN IN THE PREPARATION OF THE TECHNICAL MATERIAL PRESENTED, CONCLUSIONS DRAWN ARE NOT NECESSARILY FINAL AND MAY BE SUBJECT TO REVISION.

Technical Memorandum TM-67-77

Published by. . . . . Editorial Branch  
Technical Publications Division  
Photo/Graphics Department  
First printing . . . . . 115 copies  
Security classification . . . . . UNCLASSIFIED

## TABLE OF CONTENTS

	Page
SUMMARY .....	1
INTRODUCTION .....	3
CLASSICAL APPROACH TO ACCEPTANCE SAMPLING .....	4
Single Sample Plan .....	4
Double and Multiple Sample Plans .....	7
Sequential Sample Plan .....	9
MIL-STD-105D .....	10
DECISION THEORY APPROACH .....	10
BAYESIAN ANALYSES .....	13
Statement of Problem .....	13
Terminal Analysis .....	16
Preposterior Analysis .....	18
Numerical Example .....	20
SURVEY OF THE LITERATURE .....	20
Hamaker .....	20
Guthrie and Johns .....	22
Hald .....	22
Pfanzagl .....	24
Mazumdar .....	24
Wetherill and Campling .....	24
DISCUSSION AND SUGGESTIONS FOR FURTHER RESEARCH .....	25
BIBLIOGRAPHY .....	26
APPENDIX	
Computer Program for Terminal and Preposterior Analysis .....	29
TABLE	
Table 1. Results of Terminal and Preposterior Analysis .....	21
ILLUSTRATIONS	
Figure 1. Costs of Acceptance Sampling Plans .....	3
Figure 2. Typical OC Curve .....	6
Figure 3. OC Curve for 100-Percent Inspection .....	6
Figure 4. Parameters of OC Curve .....	7
Figure 5. OC Curves for a Double Sample Plan .....	8
Figure 6. Sequential Sample Plan Limit Lines .....	10
Figure 7. Beta Distribution With $\bar{\theta} = 0.2$ .....	14
Figure 8. Prior and Posterior Beta Distributions .....	17
Figure 9. Curve of $r_c$ for Uniform Prior Distribution and Sample Size of 10 .....	19
Figure 10. Computer Program Logic Flow Diagram .....	30

## SUMMARY

The purpose of this report is to introduce and discuss the application of statistical decision theory to the problem of acceptance testing by attributes. The classical approach to acceptance testing is introduced and discussed so that it may be contrasted with the decision theory approach. The decision theory approach, which attempts to find an optimal trade-off between the expected costs of wrong decisions and sampling costs, is illustrated by an example using the Bayesian statistical viewpoint. In the example, sample size is assumed to be predetermined and the problem is to select the optimal action based upon prior knowledge and the results of the sample inspection. The problem is then broadened to include the trade-off between the costs of wrong decisions and the costs of sampling inspection. A numerical example is solved via a simple computer program to illustrate the results of the analysis. A survey of the literature dealing with the application of decision theory to acceptance testing is presented, the contents of the report discussed, and suggestions for further work made.

## INTRODUCTION

An important step in the weapon system acquisition process is the acceptance by the government of production lots of the weapons or their subsystems. The procedures and criteria are contractually specified and depend upon statistical sampling plans in which the quality of the submitted lot is judged on the basis of a sample randomly drawn from the lot. Sampling plans have been developed by many people over a period of years and are exemplified by MIL-STD-105D, Sampling Procedures and Tables for Inspection by Attributes (reference 5), and MIL-STD-414, Sampling Procedures and Tables for Inspection by Variables for Percent Defective (reference 15). Inspection by attributes implies that the items in question are labeled as either effective or defective, while inspection by variables is based upon quantitative measurements. This paper will be restricted to the problem of inspection by attributes.

With any sampling procedure there is the possibility of making a wrong decision, that is, taking an action which would not be taken if the true quality of the entire lot were known. Each type of wrong decision, i.e., accepting a "bad" lot or rejecting a "good" lot, results in an economic disadvantage to at least one of the two parties involved. On the average, the expected costs of wrong decisions decrease as the size of the sample increases because there is less probability of a wrong decision. However, inspection itself is another economic factor, one in which costs increase with sample size. Therefore, the problem is to find an optimum trade-off between expected costs of wrong decisions and sampling costs. The situation is depicted in figure 1.

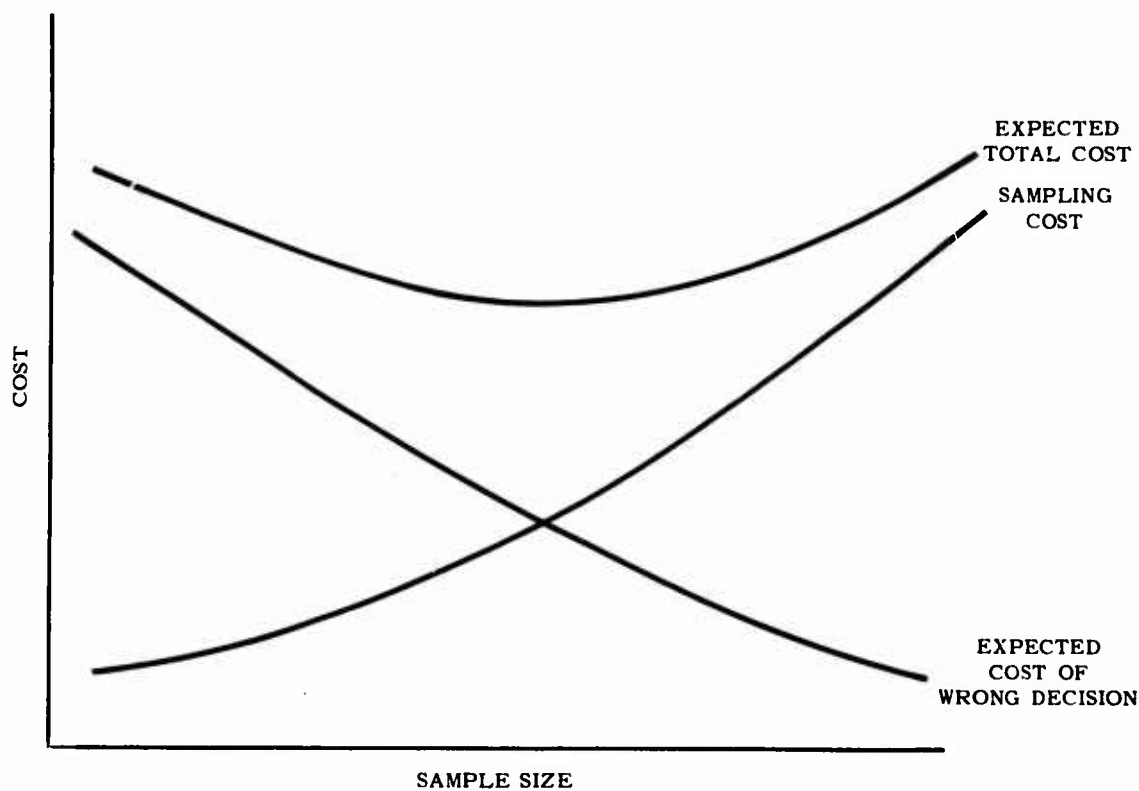


Figure 1. Costs of Acceptance Sampling Plans.

Since economic problems of the foregoing type must be resolved within a statistical framework, the general body of techniques known as statistical decision theory may be applied. The purpose of this report is to discuss the application of some of these techniques to the problem of acceptance testing.

First, the classical approach, which leads to the cited Military Standards, is reviewed so that later it may be contrasted with the decision theory approach. The standard techniques are well documented; see for example Bowker and Lieberman (reference 1) or Duncan (reference 6).

Next, the decision theory approach to acceptance testing is introduced in general terms followed by an example which illustrates the so-called Bayesian use of prior knowledge. The Bayesian viewpoint is developed in a number of books (e.g., Lindgren (reference 14) and Chernoff and Moses (reference 3)), but the primary reference is Raiffa and Schlaifer (reference 12). In the example, the sample size is first assumed to be predetermined, and the purpose of the analysis is to determine the optimal action on the basis of prior knowledge and the results of the sample inspection.

The problem is then broadened to include the trade-off between the costs of wrong decisions and the costs of sampling inspection. The procedures given permit determination of the sample size that minimizes total expected cost. A numerical example is used to illustrate the results of the analysis.

There follows a survey of the literature dealing with the application of decision theory to acceptance testing problems. The various papers are discussed, and selected assumptions and results are summarized to indicate the available procedures which may prove useful in terms of this report.

The last section briefly compares the classical and decision theory approaches to acceptance testing and suggests areas for possible further work on the problem.

This work was performed under Local Project L-2492.

## CLASSICAL APPROACH TO ACCEPTANCE SAMPLING

Consider the problem of deciding whether or not to accept a contractual lot of items, the quality of the lot being determined by the numbers of "good" and "bad" items in the lot. The lot may be accepted without inspection if the acceptor is confident that the percentage of good items is above some minimum acceptable level. Otherwise, the lot may be inspected and either accepted or rejected on the basis of the results of the inspection. An inspection of each item in the lot will determine the number of "good" and "bad" items and thus make the acceptance problem easy to solve. However, testing each individual item may be undesirable because the tests cost too much or degrade the quality of the items tested. Another alternative is to accept or reject the lot under consideration on the basis of the results of inspecting a sample randomly drawn from the lot. In general, acceptance based on sampling will prove more economical than 100-percent inspection when either the occurrence of some defectives in an accepted lot is not prohibitively expensive or when some or all of the inspection tests are destructive. In this section, various classical methods of acceptance testing will be discussed.

### *Single Sample Plan*

The most elementary acceptance sampling plan is the single sample plan. A sample of  $n$  items is randomly selected from a lot of size  $N$ . Each item in the sample is then tested and designated effective (good) or defective (bad). If the number of defective items is greater than some number,  $r_c$ , the lot is rejected; otherwise, it is accepted. This plan, characterized by the numbers  $N$ ,  $r_c$ ,  $n$  and a true lot percent defective  $\theta$ , is described by the hypergeometric probability distribution. The probability of accepting a lot with lot percent defective  $\theta$  is thus given by:

$$P\{\text{accept lot}|\theta\} = P_A = \sum_{r=0}^{r_c} \binom{\theta N}{r} \binom{N-\theta N}{n-r} / \binom{N}{n} \quad (1)$$

$$= S(N, n, \theta, r_c) \quad (2)$$

where

$\theta$  = true proportion defective in the lot

$N$  = number of items in the lot

$n$  = number of items in the sample

$r$  = observed number of defective items in the sample

$r_c$  = maximum acceptable number of defectives in sample

$$\binom{a}{b} = \frac{a!}{b!(a-b)!} \text{ for any } a, b$$

$P\{a|b\}$  means the conditional probability of  $a$ , given that  $b$  has occurred

In general, a plot of the acceptance probability as a function of  $\theta$  gives the operating characteristic (OC) curve associated with a sample plan. The ability of a particular plan to discriminate between lots of acceptable quality and lots of unacceptable quality is completely described by its associated OC curve.

In many applications the sample size  $n$  will be small compared to the lot size  $N$ , and thus the single sample plan and its OC curve may be described by the binomial probability distribution with  $n, r, \theta, r_c$  as previously defined.

$$P\{\text{accept lot}|\theta\} = P_A = P\{r \leq r_c|\theta\} = \sum_{r=0}^{r_c} \binom{n}{r} \theta^r (1-\theta)^{n-r} \quad (3)$$

$$= S(n, \theta, r_c) \quad (4)$$

Note that the plan is now independent of  $N$ , so that as long as  $n$  is small compared to  $N$ , the discrimination ability of the plan is the same regardless of the value of  $N$ . The function  $S(n, \theta, r_c)$ , when plotted, results in the OC curve shown in figure 2 for the plan  $(n, r_c)$ .

A reasonable OC curve should have a  $P_A$  of 1 when  $\theta$  is 0, and should approach 0 as  $\theta$  approaches 1. For the case of 100-percent inspection ( $n = N$ ), the OC curve assumes the simple form shown in figure 3 (where  $r_c$  is the maximum allowable number of defectives); that is,

$$P\{\text{accept lot}|\theta\} = \begin{cases} 1 & \left( \theta \leq \frac{r_c}{N} \right) \\ 0 & \left( \theta > \frac{r_c}{N} \right) \end{cases} \quad (5)$$

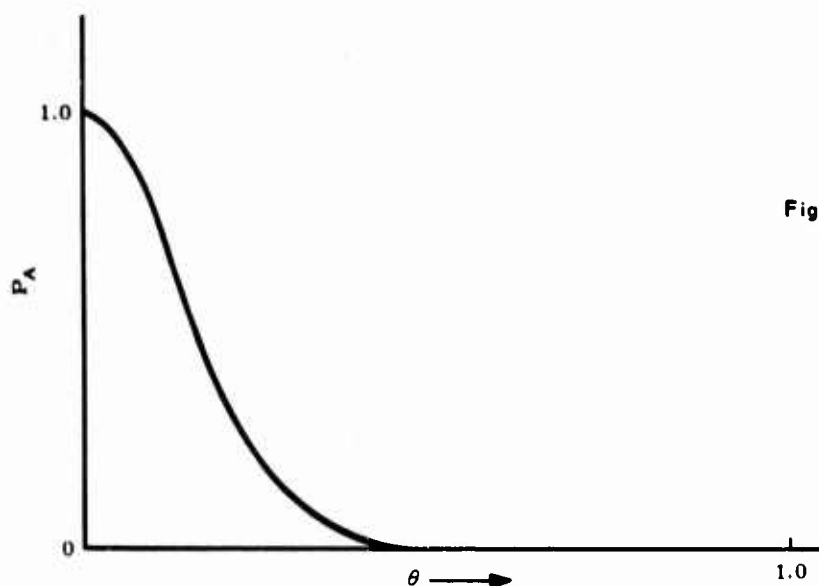


Figure 2. Typical OC Curve.

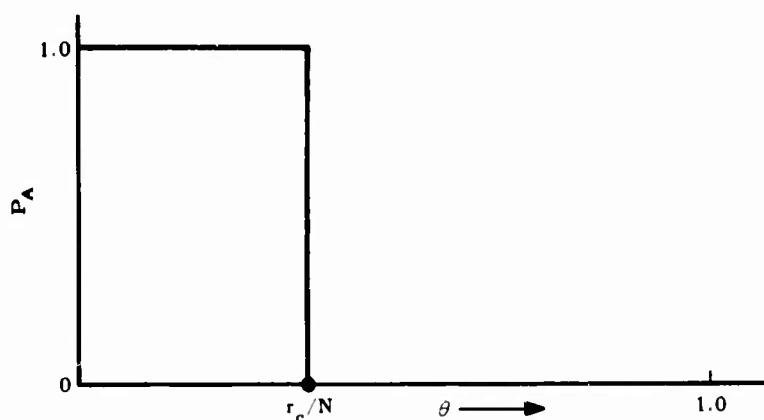


Figure 3. OC Curve for 100-Percent Inspection.

The values of  $n$  and  $r_c$  specify the distribution, equation (3), and thus the associated OC curve. Conversely, if the OC curve is specified,  $n$  and  $r_c$  may be derived. A common procedure for determining OC curves follows. With reference to figure 4, the following definitions are made:

- Producer:** Producer of the items being accepted or rejected.
- Consumer:** Party accepting or rejecting the items.
- AQL:** Acceptance quality level; indicates a high percentage of effective items for which it is desired to have a high  $P_A$ .
- LTPD:** Lot tolerance percent defective; indicates a low percentage of effective items for which it is desired to have a low  $P_A$ .
- Producers risk,  $\alpha$ :** The probability that lots where  $\theta \leq \text{AQL}$  are rejected due to vagaries of sampling. It is the probability of rejecting a lot which should have been accepted.
- Consumers risk,  $\beta$ :** The probability that lots where  $\theta \geq \text{LTPD}$  are accepted due to the vagaries of sampling. It is the probability of accepting a lot which should have been rejected.

As has been previously mentioned, the OC curve may be used to specify the parameters  $n$  and  $r_c$  of a single sample plan. The OC curve may be completely described by specifying the producer risk point and the consumer risk point. As soon as values for AQL, LTPD,  $\alpha$ , and  $\beta$  are agreed upon, equation (3) may be solved for  $n$  and  $r_c$ . An example will clarify this procedure.

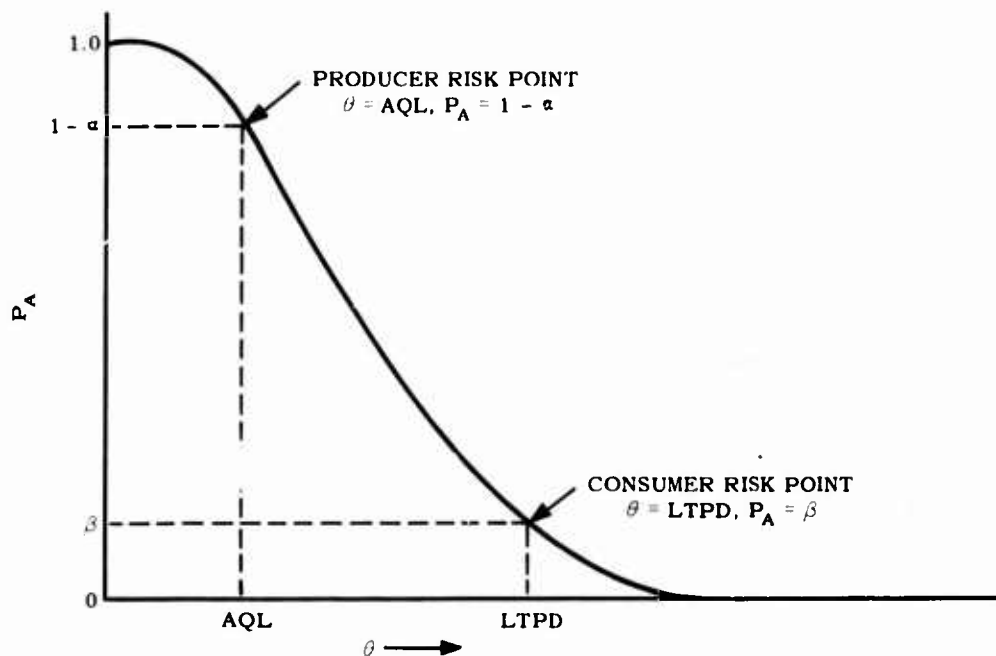


Figure 4. Parameters of OC Curve.

Example: Assume that a consumer wants to purchase items from a producer in lot quantities. The consumer desires that, on the average, lots should have a  $\theta = \text{AQL}$ , or less, proportion of defective items. The consumer does not want to accept any lots which have a  $\theta = \text{LTPD}$ , or greater proportion of defective items, but he is forced to sample test because of the prohibitive cost of 100-percent inspection. He realizes that, due to the vagaries of sampling, there is a probability that he will accept some lots for which  $\theta \geq \text{LTPD}$ , but he doesn't want to accept more than  $\beta$  of these poor quality lots. On the other hand, the producer realizes that some high quality lots with  $\theta \leq \text{AQL}$  will be rejected due to the vagaries of sampling. He is unwilling for more than  $\alpha$  of his high quality lots to be rejected. Noting that  $P_A = 1 - \alpha$  at  $\theta = \text{AQL}$ , and  $P_A = \beta$  at  $\theta = \text{LTPD}$ , substitution into equation (3) yields two equations in the two unknowns  $n$  and  $r_c$ .

$$1 - \alpha = \sum_{r=0}^{r_c} \binom{n}{r} (\text{AQL})^r (1 - \text{AQL})^{n-r} \quad (6)$$

$$\beta = \sum_{r=0}^{r_c} \binom{n}{r} (\text{LTPD})^r (1 - \text{LTPD})^{n-r}$$

These equations may be solved for  $n$  and  $r_c$ ; thus, the single sample plan has been specified by the choice of the producer and consumer risk points.

#### Double and Multiple Sample Plans

In single sample plans a decision to accept or reject a lot of items is made on the basis of evidence obtained by testing the single sample. The double sample plan allows acceptance or rejection of a lot on the basis of the results of a single sample, but also allows the alternative of examining a second sample before deciding to accept or reject the lot. Five parameters specify the double sample plan:

$N$  = lot size

$n_1$  = size of first sample

$r_{c1}$  = maximum number of defectives allowed for acceptance of lot after inspection of first sample

$n_2$  = size of second sample

$r_{c2}$  = maximum total number of defectives allowed for both samples if lot is to be accepted

Under a double sample plan, a first sample,  $n_1$ , is drawn from the lot and inspected. If  $r_{c1}$  or less defectives are observed the lot is accepted. If  $r_{c2}$  or more defectives are observed the lot is rejected. If the number of defectives observed is more than  $r_{c1}$  but less than or equal to  $r_{c2}$ , a second sample,  $n_2$ , is drawn and inspected. If the total number of defectives from the combined samples  $n_1$  and  $n_2$  is greater than  $r_{c2}$ , the lot is rejected, otherwise it is accepted. The double sample plan has the distinct advantage that lots of very high quality or lots of very low quality may be discovered by the first sample inspection, and thus fewer items may have to be inspected in order to reach an accept or reject decision. Another advantage is that lots of marginal quality may be given a second look before a decision is made. A set of typical OC curves for a double sample plan is illustrated in figure 5.

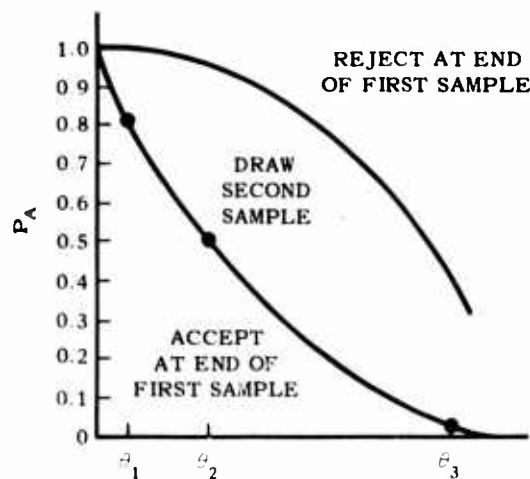
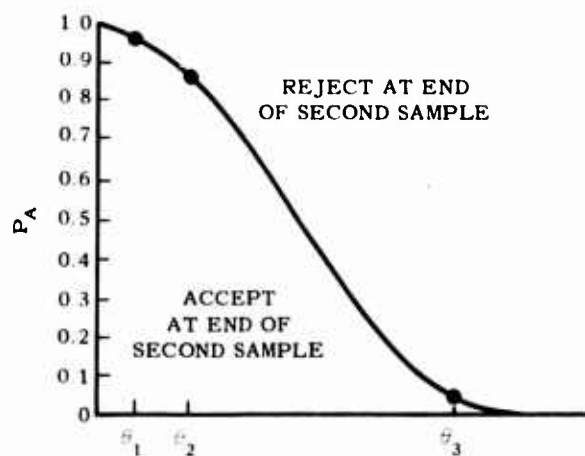


Figure 5. OC Curves for a Double Sample Plan.



Three points,  $\theta_1$ ,  $\theta_2$ , and  $\theta_3$  are chosen and it is assumed that  $\theta_1$  represents a lot of high quality,  $\theta_2$  represents a lot of acceptable, average quality, and  $\theta_3$  represents a lot of inferior, nonacceptable quality. With reference to figure 5, it can be seen that there is a high probability that a lot with a  $\theta_1$  proportion of defectives will be accepted after investigation of the first sample. There is a moderate probability that a lot of average quality with a  $\theta_2$  proportion of defectives will be accepted after investigating the first sample. The probability is high that a second sample must be drawn before a decision can be reached. The probability is also high that an inferior lot with a  $\theta_3$  proportion of defectives will be rejected after investigation of the first sample. Most of the time the decision to accept or reject the very good and very bad lots can be made on the basis of the inspection of only the first sample.

Multiple sample plans are merely extensions of the double sample plan. So if an  $m$  level sample plan is to be constructed, the parameters to be specified will be:

$N$  = lot size

$n_1, n_2, n_3, \dots, n_m$  = sizes of the  $m$  samples

$r_{c1}, r_{c2}, r_{c3}, \dots, r_{cm}$  = maximum numbers of defectives allowed for acceptance associated with the  $m$  levels

The comments made about double sample plans may be extended for multiple sample plans.

### **Sequential Sample Plan**

A sequential sample plan is the limiting case of multiple sample plans where

$$n_1 = n_2 = n_3 = \dots = n_m = 1$$

Three parameters,  $h_1$ ,  $h_2$ , and  $s$ , define the sequential plan. The parameters determine the two limit lines

$$r_c = h_2 + sn$$

$$r_c = -h_1 + sn$$

(where  $n$  is the number tested) as illustrated in figure 6. The inspection process ends as soon as one of these limit lines is reached or crossed. If, after inspection of one item, a limit line is not reached another item is inspected. This process is continued until a limit line is either reached or crossed, at which time a decision is made to accept or reject the lot under consideration.

The three parameters of the sequential sample plan,  $h_1$ ,  $h_2$ , and  $s$ , may be expressed (reference 24, Wald) in terms of the  $\alpha$ ,  $\beta$ , AQL, and LTPD of figure 4 as follows:

$$h_1 = \frac{\log [(1 - \alpha)/\beta]}{\log \{[LTPD(1 - AQL)]/[AQL(1 - LTPD)]\}}$$

$$h_2 = \frac{\log [(1 - \beta)/\alpha]}{\log \{[LTPD(1 - AQL)]/[AQL(1 - LTPD)]\}}$$

$$s = \frac{\log [(1 - AQL)/(1 - LTPD)]}{\log \{[LTPD(1 - AQL)]/[AQL(1 - LTPD)]\}}$$

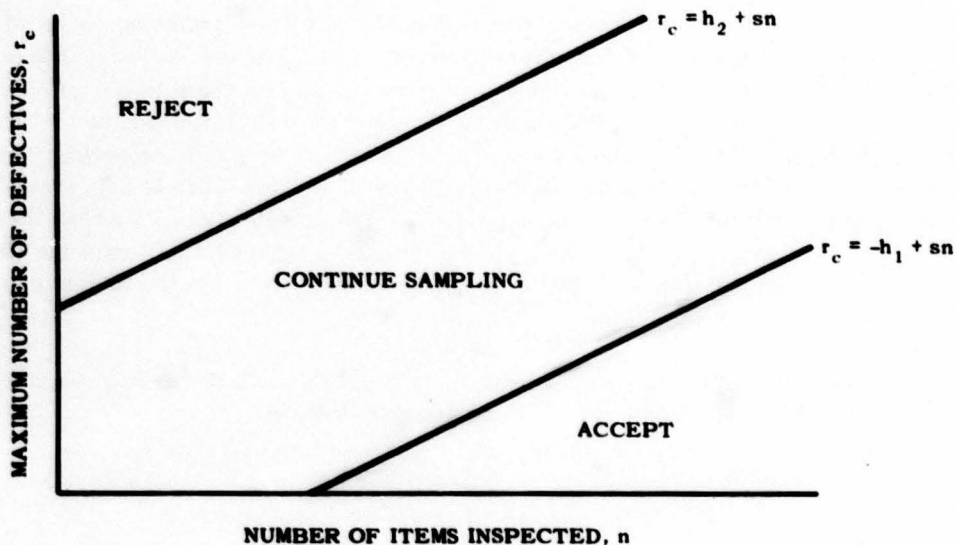


Figure 6. Sequential Sample Plan Limit Lines.

The advantages of the sequential sample plan are extensions of the advantages previously mentioned for double and multiple sample plans.

#### MIL-STD-105D

A widely used system for acceptance testing by attributes is MIL-STD-105D, reference 5, the latest revision of a system conceived in the early 1940s. The system introduced the concept of AQL, and is designed to assure a producer that his lots will be accepted a great majority of the time if they meet the AQL. Single, double, and multiple sample plans are presented with corresponding OC curves. The system specifies a sample size that increases as the lot size increases. This sample size function appears to have been chosen rather arbitrarily with the feeling that large lots of acceptable quality should run a lower risk of rejection than smaller lots meeting the AQL since there is a greater cost associated with wrongly rejecting a large lot. The system includes provisions for increasing or reducing the number of items inspected if certain criteria have been met in the inspection of previous lots. Another feature is the classification of defects as critical, major, or minor, with the provision that the number of defectives allowed depends on their severity. Interesting discussion and commentary on some of the philosophy behind the original system developed for the Army Ordnance Department is found in reference 22.

#### DECISION THEORY APPROACH

Putting acceptance testing problems within the framework of decision theory leads to a more explicit consideration of the economic consequences of accepting or rejecting a production lot as well as providing a way of combining *a priori* information and engineering judgment with data from a sampling inspection program. In this section, the basic ideas of decision theory are developed and applied to acceptance testing.

The probability model of the phenomenon, or process, being studied is of fundamental importance in the decision theory approach. It will be referred to as the process distribution. In considering a production lot in which the items can be classified into two groups, effective and defective, a natural model is the Bernoulli process, which has a probability mass function

$$f_B(x) = \theta^x(1 - \theta)^{1-x} \quad x = 0, 1$$

In this application,  $x = 0$  denotes an effective item and  $x = 1$  denotes a defective item.  $\theta$  is the probability that a randomly chosen item will be defective. The true value of  $\theta$  is unknown, although the decision maker may have some *a priori* knowledge about it. Subsequently, unknown parameter(s) will be referred to generically as the state of nature and it will be convenient to use  $\Theta$  to denote the space of all possible states of nature. For example, in the case of the parameter  $\theta$ ,  $\Theta$  could be the closed interval from 0 to 1. The state of nature will be regarded as a random variable  $\tilde{\theta}$ , the tilde being used to distinguish the random variable from a value of the random variable as denoted by  $\theta$ .

Another basic concept in decision theory is that of the set of all possible actions which the decision maker may take to "solve his problem." This space of possible actions will be indicated by  $A$  and the individual points of the space by  $a$ . For a decision about the disposition of a production lot, for example,  $A$  might have two points:  $a_1$  -- accept the lot; and  $a_2$  -- reject the lot. The correct action to take would depend upon the true state of nature, i.e., the actual proportion of defective items in the lot as denoted by  $\theta$ .

The notion of correct action requires elaboration and leads to the definition of another concept, the cost function. It will be supposed that there exists a function which indicates the monetary cost, or more generally, the utility cost, if a certain action is taken and when a certain state of nature prevails. Thus it is desirable to denote the cost function by  $C_i(\theta, a)$ , to show that the cost which is incurred depends upon the state of nature and the action taken. The reason for the  $i$  subscript will be evident later. As a simple example, suppose that for the Bernoulli process the space  $\Theta$  has only two elements,  $\theta_1$  and  $\theta_2$ , and that  $A$  has points  $a_1$  -- accept, and  $a_2$  -- reject. The cost function may be given in the form of a table, as follows:

	$\theta_1$	$\theta_2$
$a_1$	5	15
$a_2$	10	3

In the situation given here it may be supposed that  $\theta_1$  represents a small proportion of defectives and  $\theta_2$  a relatively larger proportion. Accepting the lot when the state of nature is  $\theta_2$  implies a cost of 15 units whereas accepting when the state is  $\theta_1$  implies a lesser cost of 5 units. The reject row may be interpreted in a similar way. Clearly, when the state of nature is known, there is no problem in choosing the best action; simply choose that action which minimizes the cost function. The difficulty, of course, is that the true state of nature is not known unless the entire lot is tested, a procedure which is assumed infeasible either for economic reasons or because the testing is destructive.

The procedures which may be followed to resolve the difficulty depend upon the decision maker's knowledge about the state of nature. Thus, he may claim "complete ignorance" and have no opportunity for experimentation to alleviate his ignorance. This position leads to decision criteria such as minimax, minimax regret, the principle of insufficient reason, and the Hurwicz pessimism-optimism index, discussion of which may be found in Luce and Raiffa (reference 17). Since this position does not seem applicable to the acceptance testing problem, it will not be considered further.

For the problem of acceptance testing, the relevant situation is that in which a sampling experiment may be conducted and the results used to supplement prior partial knowledge about the state of nature. However, before looking at sampling problems, it will be instructive to consider the case where no experimentation is permitted.

The prior knowledge about the state of nature can be expressed as a probability distribution on the parameter  $\theta$ . That is, the decision maker would assign weights to the various possible states of nature based upon his and other individual's experiences. The probability distribution of  $\tilde{\theta}$  will be referred to as the prior distribution, and its density function will be denoted by  $g(\theta)$ .\* A reasonable decision criterion would then be to choose that action,  $a$ , which minimizes

$$\int C_t(\theta, a) g(\theta) d\theta$$

The effect of this procedure, known as the Bayes principle, is to average the losses over the possible states of nature in each possible action. This averaging yields an expected loss for each action. The decision maker then chooses that action corresponding to the minimum expected loss.

The Bayes procedure is extended to the case where an experiment, i.e., a sampling inspection, is conducted by using the experimental results to modify the prior distribution in accordance with Bayes theorem. The new distribution of  $\tilde{\theta}$  is called the posterior distribution and its density function will be denoted by  $h(\theta|z)$  where  $z$  denotes the experimental data upon which the posterior distribution is conditioned. The new density function is given by a form of Bayes theorem:

$$h(\theta|z) = \frac{g(\theta) \ell(z|\theta)}{\ell(z)}$$

where  $\ell(z|\theta)$  is a density function called the conditional likelihood of  $z$  given  $\theta$ , and  $\ell(z)$  is the unconditional likelihood of  $z$ .

The Bayes procedure which accounts for the experimental data is then to choose that action which minimizes the posterior expected loss, i.e.,

$$\text{Min}_a \int C_t(\theta, a) h(\theta|z) d\theta$$

Since the integral is the expectation of the cost function with respect to the posterior distribution, the following notation will be used:

$$E_{\theta|z}[C_t(\tilde{\theta}, a)] = \int C_t(\theta, a) h(\theta|z) d\theta$$

For a particular experiment,  $e$ , the determination of

$$\psi(e, z) = \text{Min}_a E_{\theta|z}[C_t(\tilde{\theta}, a)]$$

for a particular  $z$  will be called the terminal analysis because it deals with the evaluation of and choice among terminal actions after the sampling inspection has been carried out. The  $t$  subscript on the cost function refers to the terminal analysis.

When the problem is broadened to include the trade-off between the costs of wrong decisions and the costs of sampling, the set of all possible experiments must be considered. This set could be denoted by  $E$  and a particular member by  $e$ . However, since only variation of sample size between experiments will be considered here, the more descriptive symbol  $n$  will be used to denote a particular member of the set. Now the cost depends upon the experiment and the cost function will

\* The tilde will be used to denote a random variable as distinguished from the generic symbol for the value of a random variable.

be denoted  $C(n, \theta, a)$ . Before the experiment is conducted then, another analysis may be performed to choose the optimum experiment, i.e., sampling plan. This analysis involves evaluating

$$\psi(n) = E_z \{ \text{Min}_a E_{\theta|z} [C(n, \hat{\theta}, a)] \}$$

for each  $n$  and choosing that  $n$  for which the expression is minimal. This second kind of analysis is called the preposterior analysis because it involves looking at the decision problem as it appears before the experiment has been conducted and taking the prior expected value of all possible posterior expected losses.

In the next section, examples are given of terminal and preposterior analyses under certain assumptions which typify an acceptance testing problem.

## BAYESIAN ANALYSES

### Statement of Problem

To illustrate the application of decision theory to acceptance testing, Bayesian analyses for a Bernoulli production process will be described. The process generates independent random variables,  $x_i$  ( $i = 1, \dots, n$ ), with identical probability mass functions

$$f_B(x) = \theta^x (1 - \theta)^{1-x} \quad x = 0, 1, \quad 0 \leq \theta \leq 1$$

As indicated previously, this is a natural model for a production process, the output of which may be classified as either effective,  $x = 0$ , or defective,  $x = 1$ .

The example will be restricted to a single sampling plan in which the sample size  $n$  is specified. This procedure, which leaves the number of defectives,  $n$ , to be determined by the experiment, is called binomial sampling. The alternative procedure, which is to have  $r$  specified and to sample until  $r$  defectives are observed, makes  $n$  a random variable and is called Pascal sampling.

The implication that Bernoulli-generated sample observations are independent is similar to the assumption in the classical approach that the binomial distribution is a good approximation for the hypergeometric. Both depend upon the lot size being large compared to the sample size.

The prior knowledge of  $\theta$  is assumed to be represented by a beta distribution, the density of which is

$$g(\theta|r', n') = f_{\beta}(\theta|r', n') = \frac{1}{\beta(r', n' - r')} \theta^{r'-1} (1 - \theta)^{n'-r'-1} \quad 0 \leq \theta \leq 1, \quad n' > r' > 0$$

with  $r'$  and  $n'$  as parameters of the distribution.  $\beta(r', n' - r')$  is the beta function defined as

$$\beta(r', n' - r') = \int_0^1 \lambda^{r'-1} (1 - \lambda)^{n'-r'-1} d\lambda$$

(where  $\lambda$  is a dummy variable of integration) or equivalently as

$$\beta(r', n' - r') = \frac{(r' - 1)! (n' - r' - 1)!}{(n' - 1)!}$$

for integer values of  $r'$  and  $n'$ . The mean of the distribution is  $\bar{\theta}' = r'/n'$ .

The choice of the beta distribution in this example is primarily for mathematical convenience\*; however, as may be inferred from figure 7, the distribution seems versatile enough to approximate a decision-maker's prior knowledge about  $\theta$ . In a real situation the choice of a specific prior distribution is apt to be largely subjective when the production contract is relatively new. However, after a number of lots have been produced, the specification of the prior distribution becomes more objective through incorporation of previous lot results via Bayes theorem. The procedure for doing this will be illustrated later. A table of percentage points of the beta distribution, which may aid the decision maker in his choice of an  $(r', n')$  pair, is provided in reference 27.

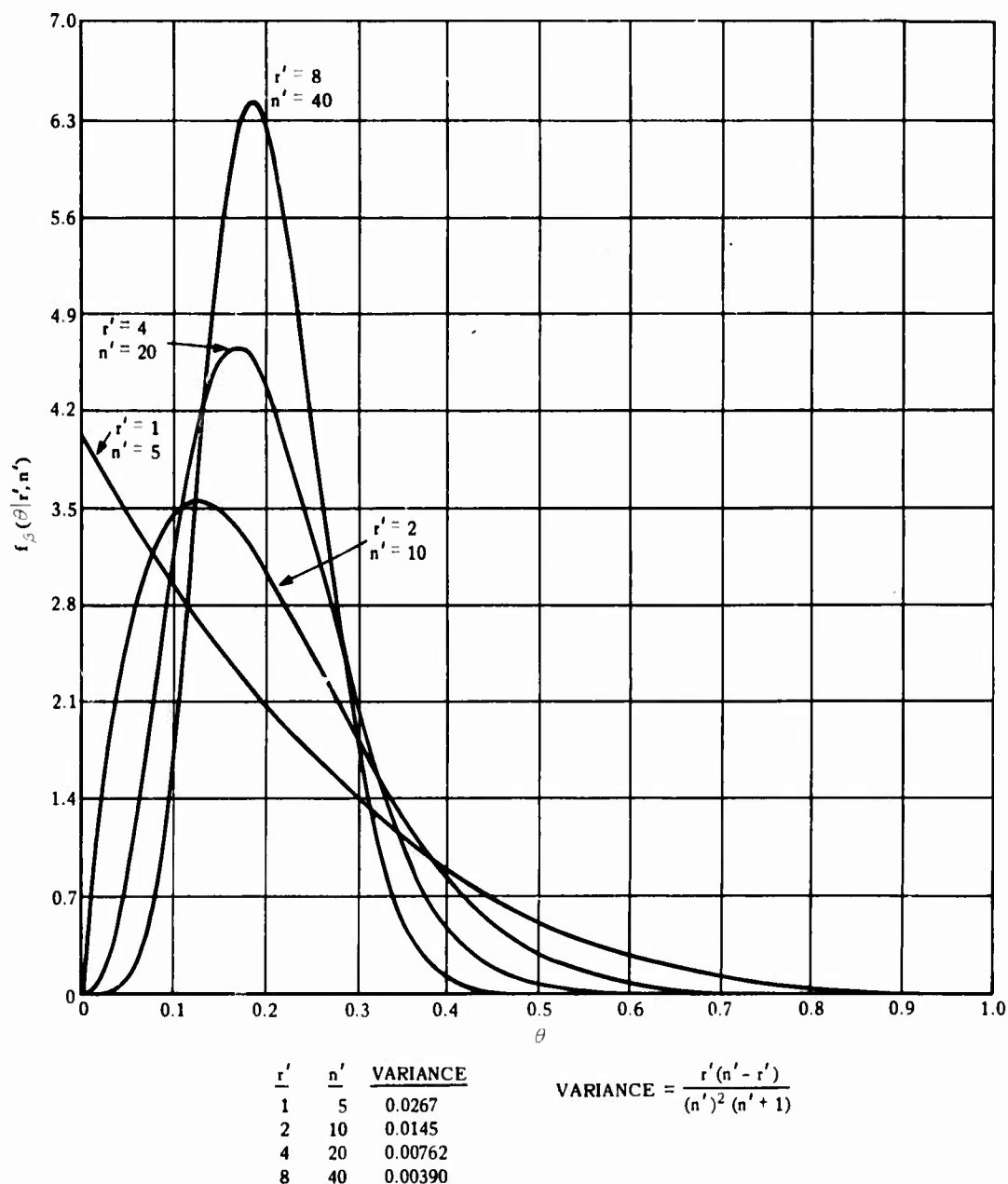


Figure 7. Beta Distribution With  $\bar{\theta} = 0.2$ .

\* The advantages which result from the choice of a beta distribution for the prior conjunction with a Bernoulli process are discussed in detail in Raiffa and Schlaifer (reference 18) and will not be repeated here. Prior distributions which have the desirable mathematical properties are referred to as natural prior conjugates by Raiffa and Schlaifer and as reproducible distributions by Hald (reference 11).

The cost function if the lot is accepted is assumed to be

$$C(n, \theta, a_1) = k_f + nk_s + \theta Nk_d$$

and if the lot is rejected is

$$C(n, \theta, a_2) = k_f + nk_s + Nk_r$$

where

$k_f$  = fixed cost of sampling, given that any sampling is done.

$k_s$  = per unit cost of sampling.

$k_r$  = per unit cost of reworking a rejected lot.

$k_d$  = per unit cost of accepting defectives.

The example may be summarized as follows:

Model: Bernoulli process with binomial sampling.

State of Nature:  $\theta$  with space  $\Theta = (\theta | 0 \leq \theta \leq 1)$ .

Action Space:  $A = (a | a_1 = \text{accept}, a_2 = \text{reject})$ .

Set of Experiments:  $E = (e | e_n = \text{experiment of sample size } n)$ .

Prior Distribution on  $\theta$ : Beta with parameters  $r'$  and  $n'$ .

Cost Function:  $C(e, \theta, a_1) = k_f + nk_s + \theta Nk_d$

$$C(e, \theta, a_2) = k_f + nk_s + Nk_r$$

Problem: (1) Terminal analysis--Given a value for  $n$ , determine the terminal action which minimizes the expected cost with respect to the posterior distribution.

(2) Preposterior analysis--Determine the sample size  $n$  which minimizes the expected cost with respect to the prior distribution.

To work the foregoing problem, one additional theoretical result will be used. It allows the experimental data, previously denoted by  $z$ , to be replaced by a sufficient statistic for these data. For example, with a single sample plan of size  $n$ ,  $z$  can be adequately described by an  $n$ -tuple,  $(x_1, x_2, \dots, x_n | x_i = 0, 1)$ , but it may also be possible to convey all *relevant* information in a more compact form, say a 2-tuple. When such is the case, the more compact form, denoted by  $y$ , is called a sufficient statistic. Roughly speaking,  $y$  is sufficient because all the relevant information regarding  $z$  is contained in  $y$ .\*

For a Bernoulli process with binomial sampling,  $y = (r, n)$ , where  $r = \sum x_i$ , is a sufficient statistic and will therefore replace  $z$  in the analysis.

Additional simplification results from a decomposition of the cost function,  $C(n, \theta, a)$ , into the sum of costs due to sampling,  $k_s + nk_s$ , and the cost due directly to the terminal action, denoted by  $C_t$ . The cost function then becomes

$$C(n, \theta, a) = k_f + nk_s + C_t(\theta, a)$$

\*The Bayesian definition of sufficiency given by Raiffa and Schlaifer (reference 18) differs from that usually found in statistics texts, but the two are equivalent.

where

$$C_t = \begin{cases} \theta Nk_d & \text{if } a = a_1 \\ Nk_r & \text{if } a = a_2 \end{cases}$$

In performing the terminal analysis for a given value of the sample size, only  $C_t$  need be considered because the sampling costs have already been expended.

### Terminal Analysis

The first step in a terminal analysis is to determine the posterior distribution  $h(\theta|y)$  corresponding to the particular  $y$  which has been observed. Such a procedure in effect updates the *a priori* knowledge about the distribution of  $\theta$  by accounting for the sample observations. Raiffa and Schlaifer (reference 18) show that the posterior distribution has the same form as the natural conjugate prior and, furthermore, that the parameters of the new distribution may be computed by a simple algebraic operation, i.e.,

$$h(\theta|y) = f_{\beta}(\theta|r'',n'') = \frac{1}{\beta(r'',n''-r'')} \theta^{r''-1} (1-\theta)^{n''-r''-1} \quad 0 \leq \theta \leq 1, \quad n'' > r'' > 0$$

where

$$r'' = r' + r$$

$$n'' = n' + n$$

Figure 8 illustrates the effect of the sample observations upon the distribution of  $\theta$ . The prior distribution for the example has parameters  $r' = 1$  and  $n' = 5$ . The mean of the distribution is  $\bar{\theta}' = r'/n' = 0.20$ .

It will be supposed that a sample of  $n = 10$  was tested and  $r = 5$  defectives were observed. The posterior distribution would then be beta with parameters  $r'' = 6$  and  $n'' = 15$  and a mean of  $\bar{\theta}'' = r''/n'' = 0.40$ .

The next step is to compute the expected terminal cost with respect to the posterior distribution for terminal actions  $a_1$  and  $a_2$ . The expected cost upon acceptance is

$$\begin{aligned} E_{\theta|y}[C_t(\tilde{\theta}, a_1)] &= \int_0^1 \theta Nk_d f_{\beta}(\theta|r'',n'') d\theta \\ &= Nk_d \bar{\theta}'' \end{aligned}$$

and the expected cost upon rejection is

$$\begin{aligned} E_{\theta|y}[C_t(\tilde{\theta}, a_2)] &= \int_0^1 Nk_r f_{\beta}(\theta|r'',n'') d\theta \\ &= Nk_r \end{aligned}$$

The ratio of the two expectations will serve as a convenient decision criterion. If

$$\frac{E_{\theta|y}[C_t(\tilde{\theta}, a_1)]}{E_{\theta|y}[C_t(\tilde{\theta}, a_2)]} = \frac{k_d}{k_r} \tilde{\theta}'' \begin{cases} > 1 & \text{reject} \\ = 1 & \text{accept or reject} \\ < 1 & \text{accept} \end{cases}$$

Since  $\tilde{\theta}'' = r''/n'' = (r' + r)/(n' + n)$ , and letting  $K = \frac{k_r}{k_d}$ , an equivalent criterion is

Reject if  $r > K(n' + n) - r'$   
 Accept or reject if  $r = K(n' + n) - r'$   
 Accept if  $r < K(n' + n) - r'$

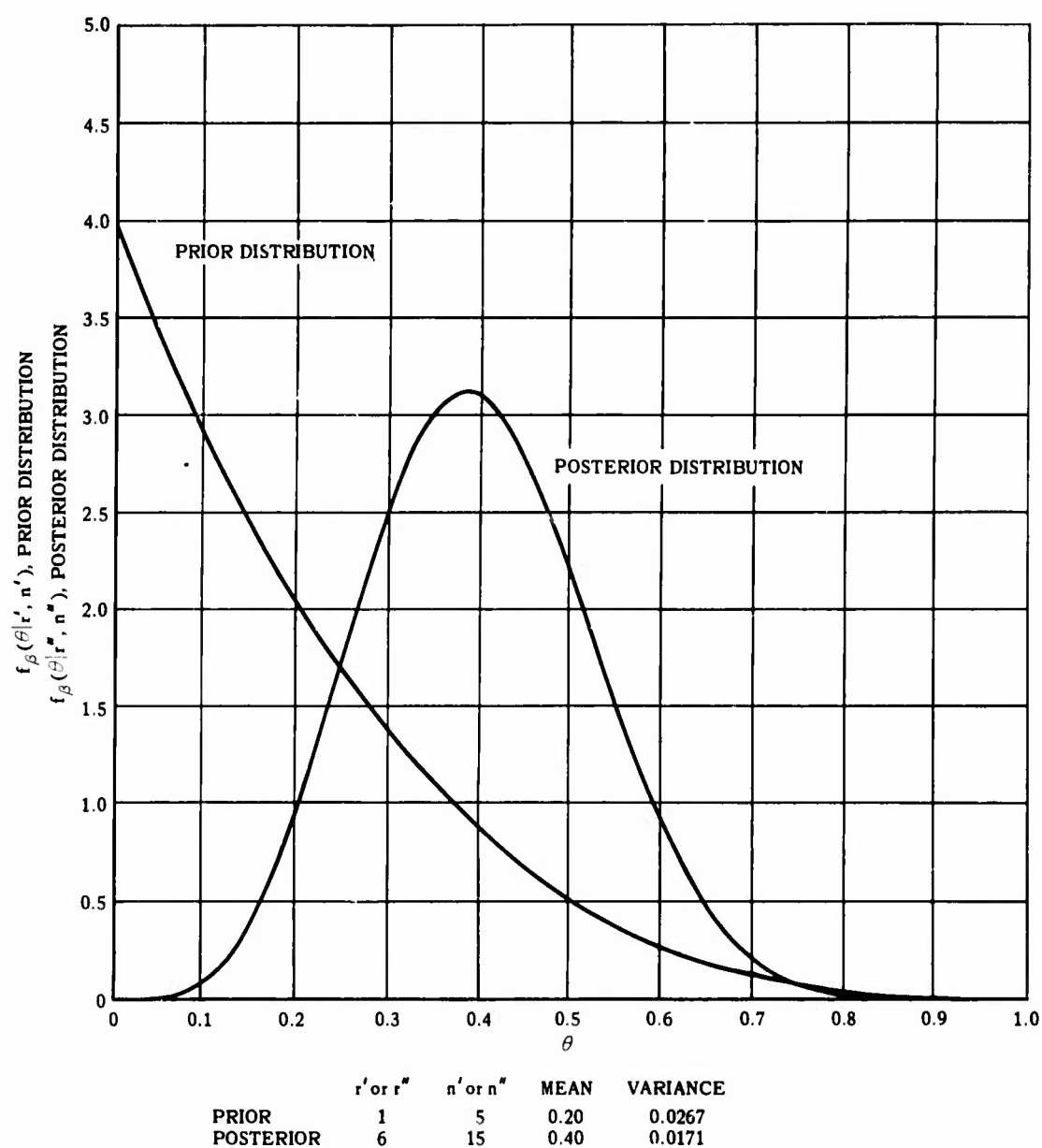


Figure 8. Prior and Posterior Beta Distributions.

The maximum acceptable number of defectives in the sample,  $r_c$ , will be the largest integer less than or equal to  $K(n' + n) - r'$ .

The special case in which the prior distribution is uniform over the range of  $\tilde{\theta}$  will now be illustrated. This case is of some interest because it implies that all values of  $\tilde{\theta}$  are considered equally likely, in a sense a situation of maximum ignorance about  $\tilde{\theta}$ . When  $n' = 2$  and  $r' = 1$

$$f_{\beta}(\theta) = 1, \quad 0 \leq \theta \leq 1$$

that is, the beta reduces to a uniform distribution. Then  $r_c$  is the largest integer less than or equal to  $K(n + 2) - 1$ . For a sample size of  $n = 10$ , the critical number  $r_c$  is given in figure 9 for values of  $K$  between 0 and 1.

If another production lot is to be tested,  $r''$  and  $n''$  serve as the parameters for new prior distribution, thus all available information is used in the new terminal analysis. It may be noted that after a number of production lots are treated in this way, the terminal action becomes relatively insensitive to the initial prior distribution parameters  $r'$  and  $n'$  and much more dependent upon the accumulated testing experience.

### Preposterior Analysis

The preposterior analysis requires the evaluation of

$$\Phi(n) = E_{y|n} \{ \text{Min}_a E_{\theta|y} [C(n, \tilde{\theta}, a)] \}$$

for each  $n$ . This involves taking the expectation of the posterior expected costs with respect to the distribution of  $\tilde{y}$ , given  $n$ , to be denoted  $f(y|n)$ . For binomial sampling from a Bernoulli process and with a beta prior, the distribution of  $\tilde{y}$ , given  $n$ , is beta-binomial (reference 18).

$$f(y|n) = f_{\beta b}(r|r', n', n) = \frac{(r + r' - 1)! (n + n' - r - r' - 1)! n! (n' - 1)!}{r! (r' - 1)! (n - r)! (n' - r' - 1)! (n + n' - 1)!}$$

for

$$r = 0, 1, 2, \dots$$

$$n = 1, 2, \dots$$

$$n \geq r$$

$$n' > r' > 0$$

If a sample is taken, the posterior expected cost with respect to  $\theta$ , given  $y$ , when the lot is accepted is

$$\begin{aligned} E_{\theta|y} [C(n, \tilde{\theta}, a_1)] &= \int_0^1 C(n, \theta, a_1) h(\theta|y) d\theta = \int_0^1 (k_f + nk_s + \theta Nk_d) f_{\beta}(\theta|r'', n'') d\theta \\ &= k_f + nk_s + \left( \frac{r' + r}{n' + n} \right) Nk_d \end{aligned}$$

and, similarly, when the lot is rejected is

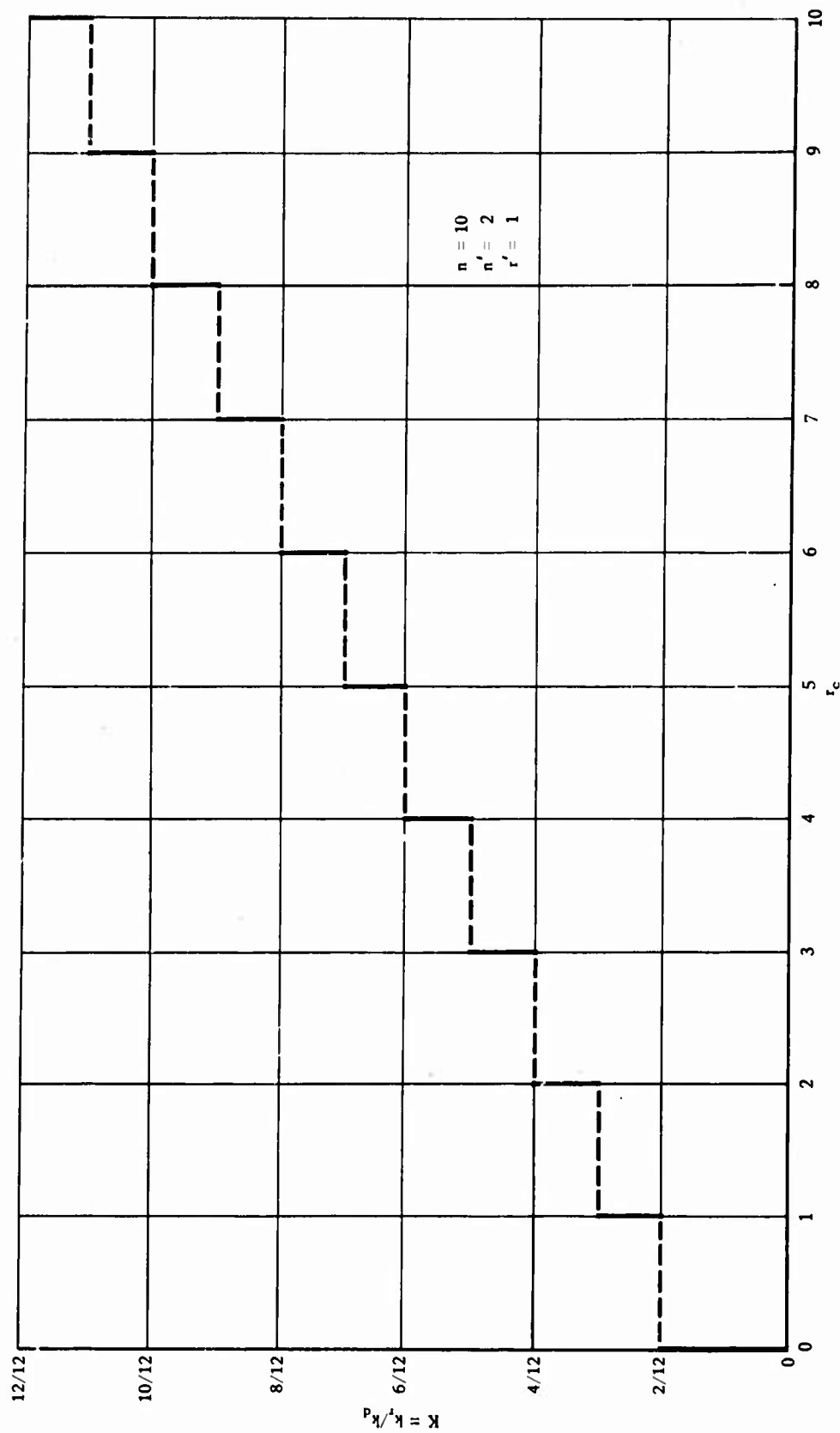


Figure 9. Curve of  $r_c$  for Uniform Prior Distribution and Sample Size of 10.

$$E_{\theta|y}[\mathcal{C}(n, \tilde{\theta}, a_2)] = k_f + nk_s + Nk_r$$

A lot is accepted when  $r$  is less than or equal to  $r_c$ ; hence, from the definition of an expectation,  $\Phi(n)$  is

$$\begin{aligned} \Phi(n) = & \sum_{r=0}^{r_c} \left[ k_f + nk_s + \left( \frac{r' + r}{n' + n} \right) Nk_d \right] f_{\beta b}(r|r', n', n) \\ & + \sum_{r=r_c+1}^n [k_f + nk_s + Nk_r] f_{\beta b}(r|r', n', n) \end{aligned}$$

for  $n = 1, 2, 3, \dots, N$ .

If the lot is accepted without sampling

$$\Phi(n) = \frac{r'}{n'} Nk_d$$

and if the lot is rejected without sampling

$$\Phi(n) = Nk_r$$

The decision maker should choose that value of  $n$  ( $= 0, \dots, N$ ) which minimizes  $\Phi(n)$ .

### Numerical Example

A FORTRAN computer program for carrying out a combined terminal and preposterior analysis is given in the appendix. The program is restricted to a beta prior distribution, binomial sampling, and linear cost functions of the type discussed in the preceding sections. Table 1 gives the parameter values and the results of the calculations.

## SURVEY OF THE LITERATURE

Some of the more interesting papers which have come to the attention of the authors and which are related to the problem of considering a decision theory approach to acceptance testing are mentioned and discussed briefly in this section. The intent is to mention those papers which are most directly applicable to the report; hence, the listing is by no means exhaustive. A relatively complete bibliography of work in this area may be generated by amalgamating the bibliographies of the references cited in this report.

Where convenient, the notation used in the mentioned papers has been changed to conform to the notation of this report.

### Hamaker

Some of the earlier work done in the area of a decision theory approach to acceptance sampling is discussed by H. Hamaker (reference 12). Hamaker presents a summary of the investigations by various authors who considered the application of economic principles to the problem of

Table 1. Results of Terminal and Preposterior Analysis

Given parameters:		
$N = 100$	$k_s = 10$	$k_d = 100$
$k_f = 5$	$k_r = 50$	$r' = 2$
Note: <i>a priori</i> failure probability = $\bar{\theta}' = 0.50$		
Sample Size, $n$	Critical Value, $r_c$	Expected Cost, $\Phi$ (Dollars)
Accept	---	5,000.00
0 Reject	---	5,000.00
1	0	4,515.00
2	1	4,525.00
3	1	4,392.14
4	2	4,402.14
5	2	4,340.71
6	3	4,350.71
7	3	4,317.42
8	4	4,327.42
9	4	4,308.29
10	5	4,318.29
11*	5	4,307.31
12	6	4,317.31
13	6	4,311.47
14	7	4,321.47
15	7	4,319.09

\* Minimum cost testing sample.

acceptance sampling. Included are the papers of Sittig (reference 20) and Weibull (reference 25). This paper is a summary of the major work in economic aspects of acceptance sampling up to its publication in 1951. It is of interest as background material, and it gives a good presentation and discussion of lot cost functions developed by four authors. These cost functions are rewritten in a uniform set of symbols so that they may be readily compared and any divergencies (and conformities) in approach may be noted and discussed.

In reference 13, H. Hamaker reviews the basic principles of sampling inspection by attributes and discusses various approaches to the problem of determining the best sample size to use in acceptance testing. Of particular interest, Hamaker discusses the selection of sample size on the basis of economic theory, which takes into account the various costs involved in sampling inspection, and summarizes and discusses particular process distributions as used by various authors. The topics of the minimax principle and the economic theory approach to acceptance testing are presented in general terms. An interesting case study is presented comparing classical acceptance testing, using the MIL-STD-105A, with Taylor's discovery sampling technique (reference 21), and the results are discussed.

### Guthrie and Johns

Guthrie and Johns (reference 9) develop single sample plans for large lots. The purpose of the paper is to find explicit asymptotic characterizations for large  $N$  of the decision procedures and sample sizes which are optimal in the Bayes sense for various classes of *a priori* probability distributions defined over the values of a parameter which denotes the true state of nature. The problem is considered for certain families of distributions of the exponential type which include the Bernoulli, the Poisson, and the Gamma as prior distributions.

The cost\* functions used are:

1. Costs of accepted lots:

$$k_d(\theta N - r) + s_1 r + k_{sn} \quad 0 \leq r \leq r_c$$

2. Costs of rejecting lots:

$$k_r(N - n) + s_1 r + k_{sn} \quad r_c + 1 \leq r \leq c$$

where

$k_d$  = cost of accepting item if it is defective (cost is 0 if effective)

$s_1$  = item cost of replacing a defective

$k_s$  = sample item inspection cost

$k_r$  = per unit cost of reworking a rejected lot

$N, n, \theta, r, r_c$  as defined before

Relations are developed which indicate the relation of sample size to increasing lot size for optimal plans.

### Hald

A. Hald (reference 11) develops a model based upon prior distributions and costs for single sample plans. Hald bases his model on the hypergeometric distribution as the process distribution instead of using a binomial or Poisson approximation to the hypergeometric as most authors have. The use of a binomial or Poisson approximation implies that the same sample size,  $n$ , and acceptance number,  $r_c$ , should be used regardless of the size of the lot (since the binomial and Poisson distribution are independent of lot size,  $N$ ). Hald points out that in most practical applications this assumption is erroneous since it disregards the economic implications of lot size. Optimum plans are developed which minimize the average costs for any given prior distribution. The hypergeometric process distribution, when weighted by the prior distribution, results in the compound hypergeometric distribution which is treated in detail. Several well-known distributions such as the hypergeometric, the Polya, the binomial, and the rectangular are special cases of the compound hypergeometric.

\*Note that these cost terms have been rewritten to conform to the notation of this report.

Hald considers two terms to represent the costs\* associated with sampling lots of quality  $\theta$ , viz

1. Costs of accepting lots:

$$nk_s + (\theta N - r)k_d \quad 0 \leq r \leq r_c \quad (i)$$

2. Costs of rejecting lots:

$$nk_s + (N - n)k_r \quad r_c + 1 \leq r \leq n \quad (ii)$$

where

$k_d$  = cost of accepting a defect

$k_s$  = item cost of sampling inspection

$k_r$  = item cost of rejected lot

$n, \theta, N, r, r_c$  as defined previously

Using equations (i), and (ii) and the hypergeometric distribution which gives the probability of getting  $r$  defects in a sample of size  $n$  drawn from a lot of size  $N$  with number of defectives  $\theta N$ ,

$$P\{r|\theta N\} = \frac{\binom{\theta N}{r} \binom{N-\theta N}{n-r}}{\binom{N}{n}} \quad (iii)$$

Hald obtains a result for the average cost per item submitted in terms of

1. Costs of sampling inspection.
2. Expected loss due to accepting bad items.
3. Cost of rejection of a lot.

The optimum sampling plan is then determined, i.e., the plan which minimizes the average cost per item submitted for any prior distribution. Hald has developed a model with which to provide answers to the following questions:

1. What savings are obtained, if any, by sampling inspection as compared with total inspection or acceptance without inspection?
2. What is the optimum relation between sample size and lot size?
3. How does the overall probability of acceptance depend on lot size?
4. How should the probability of acceptance for lots of acceptable quality increase with lot size?

In particular, an example is given presenting optimum sampling plans which answer the preceding questions for the compound binomial distribution.

\*Note that these cost terms have been rewritten to conform to the notation of this report.

Two important results are obtained for optimum sampling plans:

1. For a certain class of prior distributions which includes the Polya and rectangular distributions, the sample size increases proportional to  $\sqrt{N}$  for large  $N$ .
2. For a certain class of prior distributions which includes the binomial distribution, the sample size increases proportional to  $\log N$  for large  $N$ .

The value of the paper is further increased by accompanying discussion by other workers in the field.

#### **Pfanzagl**

J. Pfanzagl (reference 16) extends Hald's work. Pfanzagl considers a particular prior distribution, the Polya, which he claims may be conveniently used to approximate many types of empirical distributions. Using the Polya prior distribution, Pfanzagl develops double sample plans which are optimal in the sense that they minimize the average cost (risk). The effect of small changes in the prior distribution on the optimum plan is examined, and the influence of the prior distribution is only moderate. He does not examine the effects of small changes in the costs on the optimum plan. By comparison of optimum single and double sample plans, developed for some particular values of lot size costs and prior distribution, Pfanzagl concludes that in only a few cases may double sample plans be less costly than corresponding single sample plans.

#### **Mazumdar**

Pfanzagl's work is expanded to consider sequential sample plans by M. Mazumdar (reference 4). The optimal sequential sample plans are derived for the cost model used by Hald (reference 11) using a Polya prior distribution. Optimal sequential sample plans are developed, and relations for determining parameters associated with the plans such as operating characteristics, optimum boundaries for sequential procedures, terminal decision rules, acceptance and rejection numbers, etc. are given. Representative plans are developed and optimal sequential plans are compared with corresponding optimal double and single sample plans. Values of Bayes risk for the optimal single, double, and sequential plans are given for Pfanzagl's model so that a comparison of the plans may be made. The reduction in risk is shown to be small.

#### **Wetherill and Campling**

A recent paper discussing the decision theory approach to sampling inspection is presented by G. B. Wetherill and G. E. G. Campling (reference 28).

As is the custom for papers presented to the Royal Statistical Society, detailed comment, discussion, and criticism, which provide further insight into the paper accompanies the paper.

Wetherill and Campling present four different models, each describing a separate large lot sampling inspection problem and present cost functions for each model in terms of the risk.

The effects of errors on the optimum sampling plans are investigated. These errors may arise from choosing an inappropriate model to represent the problem under investigation (hence using an incorrect cost function), or from choosing an incorrect prior distribution. In addition, the effects of making bad estimates of the values of model parameters are discussed.

An investigation of the improvement expected by using double or sequential sampling plans instead of single sampling plans leads the authors to the same conclusion generally reported by other authors, namely the increase in utility is marginal.

## DISCUSSION AND SUGGESTIONS FOR FURTHER RESEARCH

Probably the most striking feature about the classical approach to acceptance sampling is the degree of arbitrariness which must accompany the choice of a plan. In the simplest case it involves the specification of the four parameters, AQL, LTPD,  $\alpha$ , and  $\beta$ , while compilations of sampling plans such as MIL-STD-105D require for most users an additional expression of faith in the underlying construction of the tables.

In the decision theory approach, acceptance testing is recognized as a problem of economic choice in the face of uncertainty. The decision-maker is performing an experiment, the purpose of which is to gain information about the production lot in question. This information has some value to the decision-maker, but it is acquired only at the cost of experimentation. The correct experiment to conduct and the correct choice between acceptance and rejection is assumed to be that which minimizes total expected cost. Application of the decision theory approach thus requires (1) the specification of cost functions relating the inspection costs and the costs of incorrect decisions and (2) a method for making use of the acquired information, e.g., the Bayesian analysis.

Though it may be granted that choice of appropriate cost functions is a difficult task, costs are more closely related to the decision process than are the parameters  $\alpha$ ,  $\beta$ , AQL, and LTPD. The choice of values for the latter is primarily a matter of convention. It is surely easier to make a rational argument to support certain cost coefficients than to show why  $\alpha$  should be 0.05 rather than, say, 0.033.

In weapons systems acceptance testing, the decision theory approach appears to offer several advantages. In particular, the Bayesian analysis offers explicit consideration of the costs involved in testing and the costs involved in introducing defects into the inventory. In addition, the information generated via the Bayesian analyses should be of value from a planning and utilization standpoint.

This paper is intended to be expository and as such offers no specific recommendations for implementation. Several problems remain to be explored. These include:

1. Before a weapons system reaches production, a large body of test data on the system and its components has been generated via contractor evaluation, preproduction tests, Navy technical evaluations, etc. Investigation of the suitability of this data for use in generating the necessary prior distributions required in developing a Bayes sampling plan should be made.
2. The costs associated with weapons systems need to be studied in order to develop meaningful cost functions to be used in developing an optimum Bayes acceptance sampling plan. These costs should include all potential costs, both direct and indirect, such as the costs of introducing defects into the inventory and the costs associated with aborting a mission because of a defect.
3. A sensitivity analysis should be made to determine how sensitive the results of the Bayesian analysis are to the form of the prior distribution, errors in the cost coefficients, and incorrect specification of the cost functions.
4. The general framework of a Bayesian sampling plan for a weapons system should be developed, possibly as a generalized computer program. This plan could then be made specific to a particular weapons system by choice of appropriate input parameters. The plan could then be checked out by using historical data on a weapons system and evaluating how well the plan would have worked.
5. In the literature (Mazumdar, Pfanzagl, Wetherill and Campling, etc.) the increase in utility achieved by double or sequential sample plans over single sample plans is shown to be marginal for the cases considered. However, the costs involved in running only one flight test

on a missile system may run into hundreds of thousands of dollars. An investigation should be made to determine if an increase in utility could be achieved by using a sequential type plan instead of a single sample plan.

6. Most of the literature has considered the case of sampling from large lots. This is not necessarily the case in acceptance testing of weapons systems. The problem of sampling from a lot where the sample size may be significant compared to the lot size should be considered from a decision theory approach.

## BIBLIOGRAPHY

1. Bowker, A. H. and G. J. Lieberman. Engineering Statistics. New Jersey, Prentice-Hall, 1959. 585 pp.
2. Bracken, J. "Percentage Points of the Beta Distribution for Use in Bayesian Analysis of Bernoulli Processes," *TECHNOMETRICS*, Vol. 8, No. 4, Nov 1966. Pp. 687-694.
3. Chernoff, H. and L. E. Moses. Elementary Decision Theory. New York, Wiley, 1959. 364 pp.
4. Cornell University College of Engineering. Optimal Sequential Plans Based on Prior Distributions and Costs, by M. Mazumdar. Ithaca, N.Y., Apr 1966. (TR-3) UNCLASSIFIED.
5. Department of Defense. Sampling Procedures and Tables for Inspection by Attributes. Washington, D.C., U.S. Government Printing Office, 29 Apr 1963. (MIL-STD-105D) UNCLASSIFIED.
6. Duncan, A. J. Quality Control and Industrial Statistics. Chicago, Richard D. Irwin, Inc., 1952. 663 pp.
7. Ellner, H. "Validating Results of Sampling Inspection by Attributes," *TECHNOMETRICS*, Vol. 5, No. 1, Feb 1963. Pp. 23-46.
8. Fabrycky, W. J. and P. E. Torgersen. Operations Economy; Industrial Applications of Operations Research. New Jersey, Prentice-Hall, 1966. 486 pp.
9. Guthrie, D. and M. V. Johns. "Bayes Acceptance Sampling Procedures for Large Lots," *ANNALS OF MATH STATISTICS*, Vol. 30. Pp. 896-925.
10. Hald, A. "Asymptotic Properties of Bayesian Single Sampling Plans," *ROY STATISTICAL SOC J*, Series B, Vol. 29, No. 1, 1967.
11. ----- "The Compound Hypergeometric Distribution and a System of Single Sampling Inspection Plans Based on Prior Distributions and Costs," *TECHNOMETRICS*, Vol. 2, No. 3, Aug 1960. Pp. 275-340.
12. Hamaker, H. C. "Economic Principles in Industrial Sampling Problems; A General Introduction," *INTERNATIONAL STATISTICAL INSTITUTE BULLETIN* 33, 1951. Pp. 105-122.
13. ----- "Some Basic Principles of Sampling Inspection by Attributes," in *Applied Statistics*, *ROY STATISTICAL SOC J*, Vol. 7, ed. by D. G. Beech. London, Oliver and Boyd, Ltd., 1958. Pp. 149-159.
14. Lindgren, B. W. Statistical Theory. New York, MacMillan, 1962. 427 pp.
15. Department of Defense. Sampling Procedures and Tables for Inspection by Variables for Percent Defective. Washington, D.C., U.S. Government Printing Office, 11 Jun 1957. (MIL-STD-414) UNCLASSIFIED.
16. Pfanzagl, J. "Sampling Procedures Based on Prior Distributions and Costs," *TECHNOMETRICS*, Vol. 5, No. 1, Feb 1963. Pp. 47-61.
17. Raiffa, H. and R. D. Luce. Games and Decisions; Introduction and Critical Survey. New York, Wiley, 1957. 509 pp.
18. Raiffa, H. and R. Schlaifer. Applied Statistical Decision Theory. Boston, Harvard University Division of Research, Graduate School of Business Administration, 1961. 356 pp.
19. Schlaifer, R. Introduction to Statistics for Business Decisions. New York, McGraw-Hill, 1961. 382 pp.
20. Sittig, J. "The Economic Choice of Sampling System in Acceptance Sampling," *INTERNATIONAL STATISTICAL INSTITUTE BULLETIN* 33, 1951. Pp. 51-83.

21. Taylor, E. F. "Discovery Sampling by Attributes," NATIONAL CONVENTION TRANSACTIONS of the American Society for Quality Control, 1957.
22. The American Statistical Association. "Acceptance Sampling" presented at the 105th Annual Meeting at Cleveland, Ohio, 27 Jan 1946.
23. Tippett, L. "A Guide to Acceptance Sampling," in Applied Statistics, ROY STATISTICAL SOC J, Vol. 7, ed. by D. G. Beech. London, Oliver and Boyd, Ltd., 1958. Pp. 133-148.
24. Wald, A. Sequential Analysis. New York, Wiley, 1947. 212 pp.
25. Weibull, I. "A Method of Determining Inspection Plans on an Economic Basis," INTERNATIONAL STATISTICAL INSTITUTE BULLETIN 33, 1951. Pp. 85-104.
26. Wetherill, G. B. Sequential Methods in Statistics. New York, Wiley, 1966. 218 pp.
27. -----, "Some Remarks on the Bayesian Solution of the Single Sample Inspection Scheme," TECHNOMETRICS, Vol. 2, No. 3, Aug 1960. Pp. 341-352.
28. Wetherill, G. B. and G. E. G. Campling. "The Decision Theory Approach to Sampling Inspection," ROY STATISTICAL SOC J, Vol. 28, No. 3, 1966. Pp. 381-416.

## APPENDIX

### COMPUTER PROGRAM FOR TERMINAL AND PREPOSTERIOR ANALYSIS

The following program may be used to carry out a Bayesian analysis of an acceptance testing problem with a linear cost function, a beta prior distribution, and a binomial sampling plan. The program includes a preposterior analysis for determining the optimal sample size and a terminal analysis for determining the optimal decision rule, given the sample size.

Notation:

Math Model	Computer Program	Definition
$n$	NSAM	Sample size
$N$	N	Lot size
	NLIMIT	Limit of sample size (to reduce amount of computation)
$k_f$	FCOST	Fixed cost
$k_s$	SCOST	Sampling cost
$k_r$	RCOST	Rework cost
$k_d$	DCOST	Defective cost
$r'$	RPRIME	} <i>a priori</i> distribution parameters
$n$	NPRIME	
$r$	DEFECT	Number of defectives
$\Phi$	PHI	Total expected cost
$r_c$	RCRIT	Critical value for a sampling plan
$C(n, \tilde{\theta}, a_1)$	COST1	Posterior expected cost, given acceptance
$C(n, \tilde{\theta}, a_2)$	COST2	Posterior expected cost, given rejection
$f_{\beta b}$	PROB	Probability mass function for beta-binomial distribution

The logic flow diagram for the computer program is shown in figure 10. The computer program listing, written in FORTRAN IV, follows.

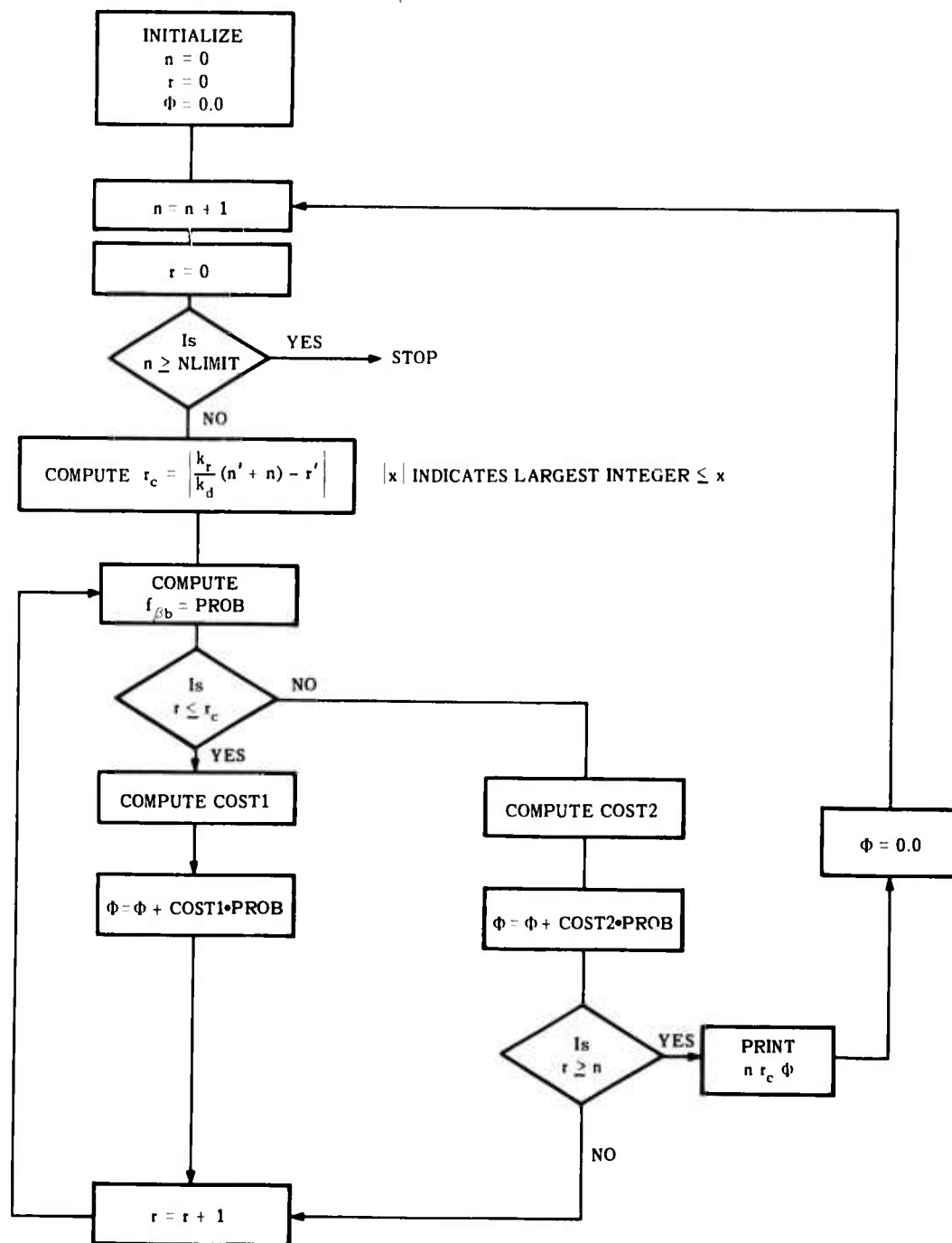


Figure 10. Computer Program Logic Flow Diagram.

```

MAIN PROGRAM
INTEGER DEFECT,RPRIME,RCRIT
WRITE (6,2)
2 FORMAT (1H1,79H N NLIMIT FIXEDCOST SAMPLINGCOST REWORKCOST
XDEFECTCOST RPRIME NPRIME )
READ(5,3) N,NLIMIT,FCOST,SCOST,RCOST,DCOST,RPRIME,NPRIME

3 FORMAT(2I10,4F10.2,2I10)
WRITE(6,4)N,NLIMIT,FCOST,SCOST,RCOST,DCOST,RPRIME,NPRIME
4 FORMAT(14,15,F14.2,F14.2,2F12.2,16,18)
WRITE(6,5)
5 FORMAT(1H0,45H SAMPLE SIZE CRITICAL VALUE EXPECTED LOSS )
NSAM=0
DEFECT=0
PHI=0.)
6 NSAM=NSAM+1
DEFECT = 0
IF (NSAM.GT.NLIMIT) GO TO 100
REALR=((RCOST/DCOST)*FLOAT(NPRIME+NSAM))-FLOAT(RPRIME)
RCRIT=IFIX(REALR)
COMPUTE BETA-BINOMIAL PROBABILITY
10 PROB=(FACT(DEFECT+RPRIME-1)*FACT(NSAM+NPRIME-DEFECT-RPRIME-1)*FACT
X(NSAM)*FACT(NPRIME-1))/(FACT(DEFECT)*FACT(RPRIME-1)*FACT(NSAM-DEFE
XCT)*FACT(NPRIME-RPRIME-1)*FACT(NSAM+NPRIME-1))
IF (DEFECT.GT.RCRIT) GO TO 50
COMPUTE ACCEPTANCE COST
40 COST1=FCOST+FLOAT(NSAM)*SCOST+(((FLOAT(RPRIME)+FLOAT(DEFECT))/(FLOA
XT(NPRIME)+FLOAT(NSAM)))*FLOAT(N)*DCOST
COMPUTE ONE TERM OF EXPECTED COST AND ADD TO PREVIOUS TERMS
PHI=PHI+(COST1*PROB)
GO TO 30
50 CONTINUE
60 COST2=FCOST+FLOAT(NSAM)*SCOST+FLOAT(N)*RCOST
PHI=PHI+(COST2*PROB)
IF (DEFECT.LT.NSAM) GO TO 80
65 WRITE(6,70) NSAM,RCRIT,PHI
70 FORMAT(16,114,F23.2)
PHI=0.)
GO TO 5
80 DEFECT=DEFECT+1
IF (DEFECT.GT.NSAM) GO TO 65
GO TO 10
100 CONTINUE
STOP
END

```

```

SUBPROGRAM TO COMPUTE THE FACTORIAL OF M
FUNCTION FACT(M)
Y=1.0
DO 1 I=1,M
Y=Y*FLOAT(I)
1 CONTINUE
FACT = Y
RETURN
END

```

UNCLASSIFIED

Security Classification

DOCUMENT CONTROL DATA - R & D		
<i>Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified</i>		
1. ORIGINATING ACTIVITY (Corporate author)		2a. REPORT SECURITY CLASSIFICATION
Naval Missile Center Point Mugu, California		UNCLASSIFIED
		2b. GROUP
		---
3. REPORT TITLE		
A DECISION THEORY APPROACH TO ACCEPTANCE TESTING		
4. DESCRIPTIVE NOTES (Type of report and inclusive dates)		
5. AUTHOR(S) (First name, middle initial, last name)		
E. D. Simmons and C. E. Wisler		
6. REPORT DATE	7a. TOTAL NO. OF PAGES	7b. NO. OF REFS
31 January 1968	31	28
8a. CONTRACT OR GRANT NO.	9a. ORIGINATOR'S REPORT NUMBER(S)	
b. PROJECT NO. Local Project L-2492	TM-67-77	
c.	9b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report)	
d.		
10. DISTRIBUTION STATEMENT		
This document has been approved for public release and sale; its distribution is unlimited.		
11. SUPPLEMENTARY NOTES		12. SPONSORING MILITARY ACTIVITY
		Naval Air Systems Command
13. ABSTRACT		
<p>The purpose of this report is to introduce and discuss the application of statistical decision theory to the problem of acceptance testing by attributes. The classical approach to acceptance testing is introduced and discussed so that it may be contrasted with the decision theory approach. The decision theory approach, which attempts to find an optimal trade-off between the expected costs of wrong decisions and sampling costs, is illustrated by an example using the Bayesian statistical viewpoint. In the example, sample size is assumed to be predetermined and the problem is to select the optimal action based upon prior knowledge and the results of the sample inspection. The problem is then broadened to include the trade-off between the costs of wrong decisions and the costs of sampling inspection. A numerical example is solved via a simple computer program to illustrate the results of the analysis. A survey of the literature dealing with the application of decision theory to acceptance testing is presented, the contents of the report discussed, and suggestions for further work made.</p>		

DD FORM 1 NOV 65 1473 (PAGE 1)  
S/N 0101-807-6801

UNCLASSIFIED  
Security Classification

**Security Classification**

14	KEY WORDS	LINK A		LINK B		LINK C	
		ROLE	WT	ROLE	WT	ROLE	WT
	Decision theory Bayesian statistics Sampling theory Acceptance testing						

DD FORM 1473 (BACK)  
(PAGE 2)

**UNCLASSIFIED**  
Security Classification